
3rd edition

Sheldon Axler

# PRECALCULUS

## A PRELUDE TO CALCULUS

with Student Solutions Manual

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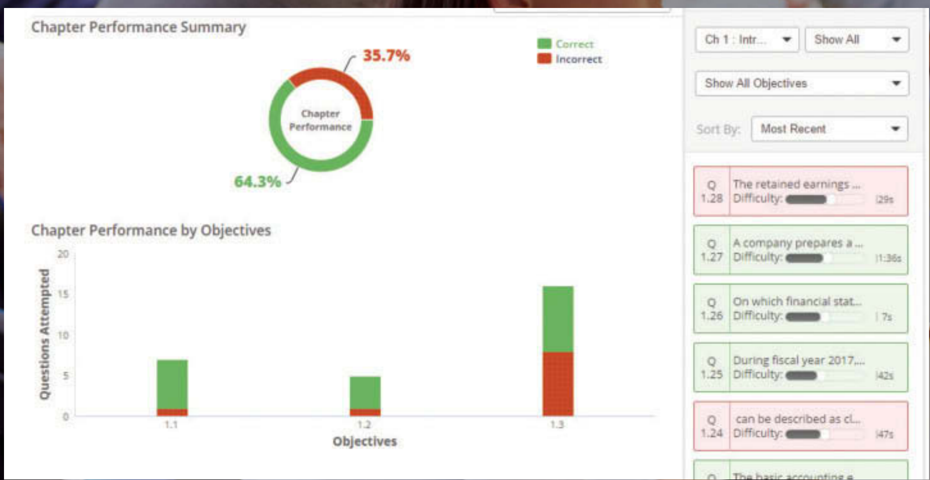


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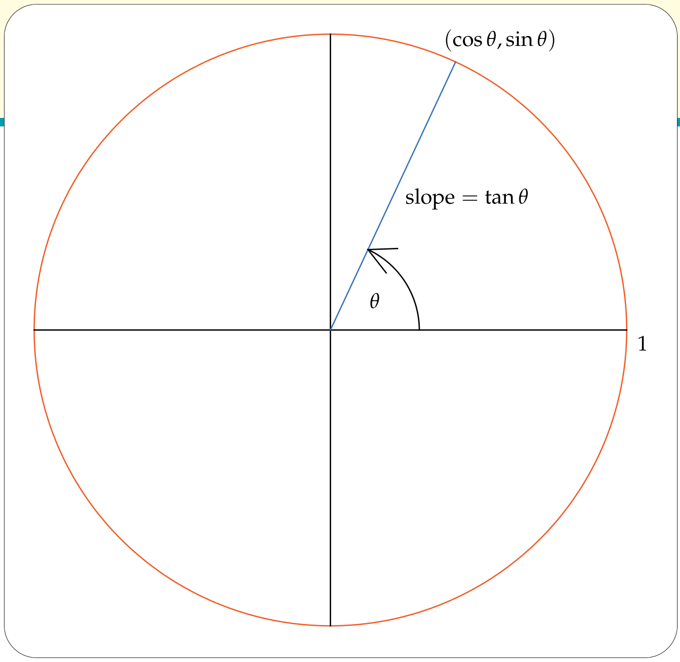


# *Precalculus*

## *A Prelude to Calculus*

Third edition

*with*
Student Solutions Manual


Sheldon Axler
*San Francisco State University*



WILEY

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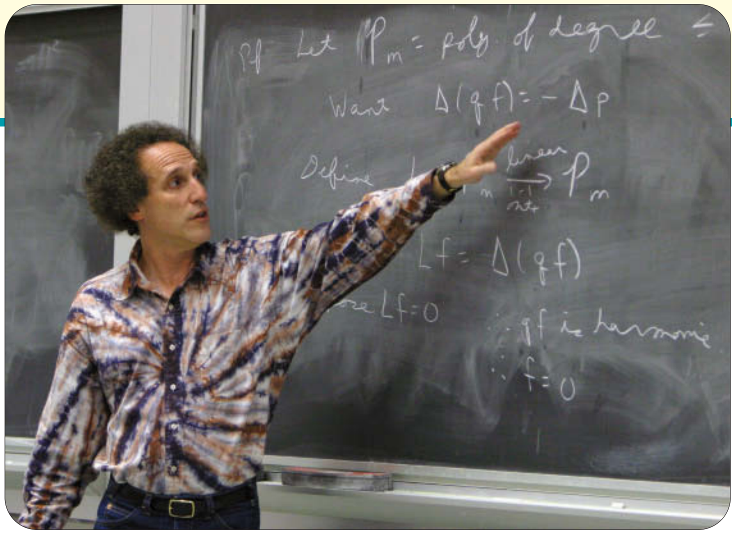
# About the Author



*Sheldon Axler*

Sheldon Axler was valedictorian of his high school in Miami, Florida. He received his AB from Princeton University with highest honors, followed by a PhD in Mathematics from the University of California at Berkeley.

As a Moore Instructor at MIT, Axler received a university-wide teaching award. He was then an assistant professor, associate professor, and professor in the Mathematics Department at Michigan State University, where he received the first J. Sutherland Frame Teaching Award and the Distinguished Faculty Award.

Axler came to San Francisco State University as Chair of the Mathematics Department in 1997. In 2002, he became Dean of the College of Science & Engineering at SF State, a position he held until returning full-time to mathematics in 2015.

Axler received the Lester R. Ford Award for expository writing from the Mathematical Association of America in 1996. In addition to publishing numerous research papers, Axler is the author of five mathematics textbooks, ranging from freshman to graduate level. His book *Linear Algebra Done Right* has been adopted as a textbook at over 300 universities.

Axler has served as Editor-in-Chief of the *Mathematical Intelligencer* and as Associate Editor of the *American Mathematical Monthly*. He has been a member of the Council of the American Mathematical Society and a member of the Board of Trustees of the Mathematical Sciences Research Institute. Axler currently serves on the editorial board of Springer's series Undergraduate Texts in Mathematics, Graduate Texts in Mathematics, and Universitext.

The American Mathematical Society honored Axler by selecting him as a member of its inaugural group of Fellows in 2013.

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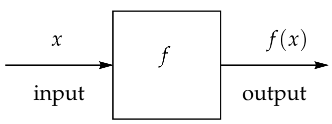
**About the Cover**

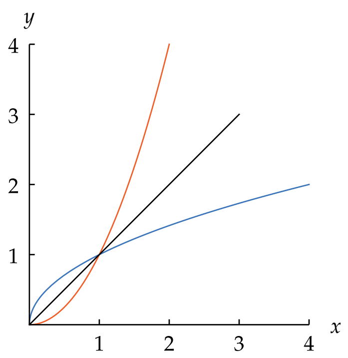
The diagram on the cover contains the crucial definitions of trigonometry.

- The 1 shows that the trigonometric functions are defined in the context of the unit circle.
- The arrow shows that angles are measured counterclockwise from the positive horizontal axis.
- The point labeled $(\cos\theta, \sin\theta)$ shows that $\cos\theta$ is the first coordinate of the endpoint of the radius corresponding to the angle $\theta$, and $\sin\theta$ is the second coordinate of this endpoint. Because this endpoint is on the unit circle, the identity $\cos^2\theta + \sin^2\theta = 1$ follows immediately.
- The equation "slope $=$ $\tan\theta$" shows that $\tan\theta$ is the slope of the radius corresponding to the angle $\theta$; thus $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

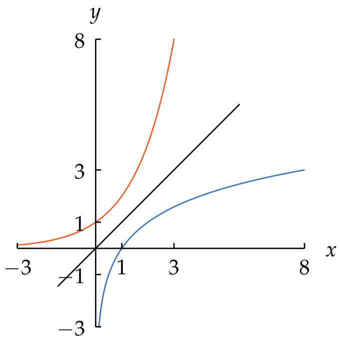
# Contents





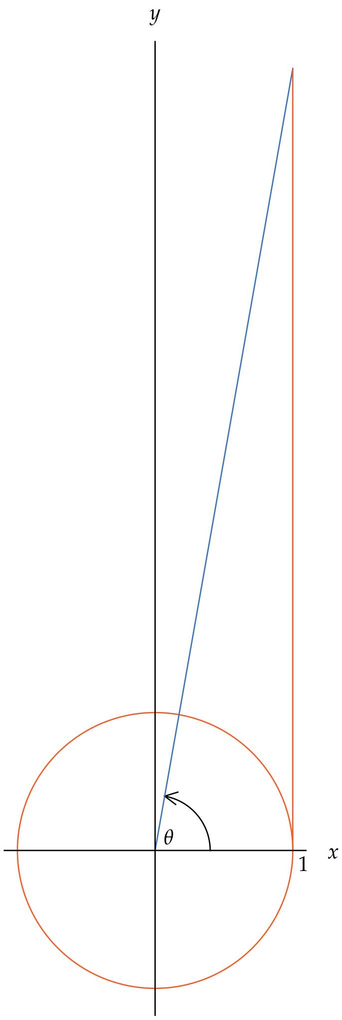



*The graphs of $x^2$ (orange) on $[0,2]$ and its inverse function $\sqrt{x}$ (blue) on $[0,4]$ are symmetric about the line $y = x$.*



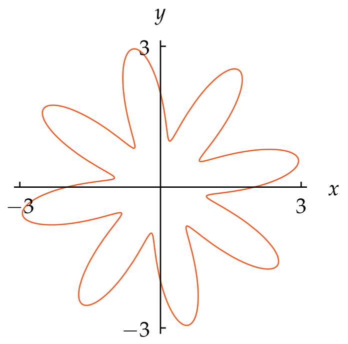

*The graphs of $2^x$ (orange) on $[-3,3]$ and its inverse function $\log_2 x$ (blue) on $[\frac{1}{8},8]$ are symmetric about the line $y=x$.*

*Starry Night, painted by Vincent Van Gogh in 1889. The brightness of a star as seen from Earth is measured using a logarithmic scale.*




*The blue line segment has slope* $\tan\theta$. *The orange line segment has length* $\tan\theta$.

*The Greek mathematician Hipparchus, depicted here in a $19^{th}$-century illustration, developed trigonometry over 2100 years ago as a tool for calculations in astronomy.*




*The graph of the polar equation* $r = 2 + \sin(8\theta)$ *for* $\theta$ *in* $[0, 2\pi]$.

# Preface to the Instructor

## Goals and Prerequisites

This book seeks to prepare students to succeed in calculus. Thus it focuses on topics that students need for calculus, especially first-semester calculus. Important parts of mathematics that should be known by all educated citizens but that are irrelevant to calculus have been excluded.

Precalculus is a one-semester course at most colleges and universities. Nevertheless, typical precalculus textbooks contain about a thousand pages (not counting a student solutions manual), far more than can be covered in one semester.

By emphasizing topics crucial to success in calculus, this book has a more manageable size even though it includes a student solutions manual. A thinner textbook should indicate to students that they are truly expected to master most of the content of the book.

The prerequisite for this course is the usual course in intermediate algebra. Many students in precalculus classes have had a trigonometry course previously, but this book does not assume students remember any trigonometry. The book is fairly self-contained, starting with a review of the real numbers in Chapter 0, whose numbering is intended to indicate that many instructors will prefer to cover this beginning material quickly or skip it.

*Chapter 0 could have been titled* **A Prelude to A Prelude to Calculus**.

Different instructors will want to cover different sections of this book. My personal preference is to finish up through Section 6.2 (*Series*). By including the sections on sequences and series, you will give students some experience with using subscript notation and summation notation that will be useful when they get to Riemann integration. The last chapter (*Polar Coordinates, Vectors, and Complex Numbers*) deals with topics that are typically more useful for second-semester calculus. I do not cover this chapter when teaching precalculus because I prefer to focus on getting students ready to succeed in first-semester calculus. Other instructors have different preferences, which is why I have included the last chapter in this book.

## A Book Designed to be Read

Mathematics faculty frequently complain, with justification, that most students in lower-division mathematics courses do not read the textbook.

When doing homework, a typical precalculus student looks only at the relevant section of the textbook or the student solutions manual for an example similar to the homework exercise at hand. The student reads enough of that example to imitate the procedure, does the homework exercise, and then follows the same process with the next homework exercise. Little understanding may take place.

In contrast, this book is designed to be read by students. The writing style and layout are meant to induce students to read and understand the material. Explanations are more plentiful than typically found in precalculus books, with examples of the concepts making the ideas concrete whenever possible.

*The text often points out to students that understanding the material will be more useful than memorizing it.*

As a visual aid to students, boxes in this book are color-coded to show their function. Specifically, boxes with yellow shading give definitions, and boxes with blue shading give results (which in many books are called theorems or corollaries).

## Exercises and Problems

*Each exercise in this book has a unique correct answer, usually a number or a function. Each problem in this book has multiple correct answers, usually consisting of explanations or examples.*

Students learn mathematics by actively working on a wide range of exercises and problems. Ideally, a student who reads and understands the material in a section of this book should be able to do the exercises and problems in that section without further help. However, some of the exercises require application of the ideas in a context that students may not have seen before, and many students will need help with these exercises. This help is available from the complete worked-out solutions to all the odd-numbered exercises that appear at the end of each section.

Because the worked-out solutions were written solely by the author of the textbook, students can expect an unusually consistent approach to the material. Students will be happy to save money by not having to purchase a separate student solutions manual.

*This book contains what is usually a separate book called the student solutions manual. Thus it is even thinner in comparison to competing bloated books than is indicated by just a page count.*

The exercises (but not the problems) occur in pairs, so that an odd-numbered exercise is followed by an even-numbered exercise whose solution uses the same ideas and techniques. A student stumped by an even-numbered exercise should be able to tackle it after reading the worked-out solution to the corresponding odd-numbered exercise. This arrangement allows the text to focus more centrally on explanations of the material and examples of the concepts.

Many students read the student solutions manual when they are assigned homework, even though they may be reluctant to read the main text. The integration of the student solutions manual within this book may encourage students who would otherwise read only the student solutions manual to drift over and also read the main text. To reinforce this tendency, the worked-out solutions to the odd-numbered exercises at the end of each section are typeset in a slightly less appealing style (smaller type and two-column format) than the main text. The reader-friendly appearance of the main text may nudge students to spend some time there.

Exercises and problems in this book vary greatly in difficulty and purpose. Some exercises and problems are designed to hone algebraic manipulation skills; other exercises and problems are designed to push students to genuine understanding beyond rote algorithmic calculation.

Some exercises and problems intentionally reinforce material from earlier in the book. For example, Exercise 27 in Section 4.3 asks students to find the smallest number $x$ such that $\sin(e^x) = 0$; students will need to understand that they want to choose $x$ so that $e^x = \pi$ and thus $x = \ln \pi$. Although such exercises require more thought than most exercises in the book, they allow students to see crucial concepts more than once, sometimes in unexpected contexts.

For instructors who want to offer online grading to their students, exercises from this book are available via either *WileyPLUS* or *WebAssign*. These online learning systems give students instant feedback and keep records for instructors. Most of the exercises in this book have been translated into algorithmically generated exercises in these two online learning systems, creating an essentially unlimited number of variations. These systems give instructors the flexibility of allowing students who answer an exercise incorrectly to attempt similar exercises requiring the same ideas and techniques.

## The Calculator Issue

*To aid instructors in presenting the kind of course they want, the symbol appears with exercises and problems that require students to use a calculator.*

The issue of whether and how calculators should be used by students has generated immense controversy.

Some sections of this book have many exercises and problems designed for calculators—examples include Section 3.4 on exponential growth and Section 5.4 on the law of sines and the law of cosines. However, some sections deal with material not as amenable to calculator use. Throughout the text, the emphasis is on giving students both the understanding and the skills they need to succeed in calculus. Thus the book does not aim for an artificially predetermined percentage of exercises and problems in each section requiring calculator use.

Some exercises and problems that require a calculator are intentionally designed to make students realize that by understanding the material, they can overcome the limitations of calculators. For example, Exercise 15 in Section 3.2 asks students to find the number of digits in the decimal expansion of $7^{4000}$. Brute force with a calculator will not work with this problem because the number involved has too many digits. However, a few moments' thought should show students that they can solve this problem by using logarithms (and their calculators!).

The calculator icon  can be interpreted for some exercises, depending on the instructor's preference, to mean that the solution should be a decimal approximation rather than the exact answer. For example, Exercise 3 in Section 3.7 asks how much would need to be deposited in a bank account paying 4% interest compounded continuously so that at the end of 10 years the account would contain \$10,000. The exact answer to this exercise is $10000/e^{0.4}$ dollars, but it may be more satisfying to the student (after obtaining the exact answer) to use a calculator to see that approximately \$6,703 needs to be deposited. For such exercises, instructors can decide whether to ask for exact answers or decimal approximations (the worked-out solutions for the odd-numbered exercises usually contain both types of solutions).

*Regardless of what level of calculator use an instructor expects, students should not turn to a calculator to compute something like* $\cos 0$, *because then* cos *has become just a button on the calculator.*

## Functions

In preparation for writing this book, I asked many calculus instructors what improvements they would like to see in the preparation of their calculus students. The two most common answers I received were (1) better understanding of functions and (2) better algebraic manipulation skills. Both of these goals are intertwined throughout the book.

Because of the importance of functions, Chapter 1 (*Functions and Their Graphs*) is devoted to functions, considerably earlier than in many precalculus books. Particular attention is paid to function transformations, composition of functions, and inverse functions.

*A good understanding of the composition of functions will be tremendously useful to students when they get to the chain rule in calculus.*

The unifying concept of inverse functions appears several times later in the book. In particular, $y^{1/m}$ is defined as the number that when raised to the $m^{\text{th}}$ power gives $y$ (in other words, the function $y \mapsto y^{1/m}$ is the inverse of the function $x \mapsto x^m$; see Section 2.3). Later, a second major use of inverse functions occurs in the definition of $\log_b y$ as the number such that $b$ raised to this number gives $y$ (in other words, the function $y \mapsto \log_b y$ is the inverse of the function $x \mapsto b^x$; see Section 3.1).

Thus students should be comfortable with using inverse functions by the time they reach the inverse trigonometric functions (arccosine, arcsine, and arctangent) in Section 5.1. This familiarity with inverse functions should help students deal with inverse operations (such as antidifferentiation) when they reach calculus.

Chapter 2 (*Linear, Quadratic, Polynomial, and Rational Functions*) should be mostly review of what students learned in their intermediate algebra course. I placed the more demanding Chapter 1 first because there is a serious danger of boring students in a precalculus class if they develop an early feeling that they already know all this material.

## Logarithms, $e$, and Exponential Growth

The base for logarithms in Chapter 3 is arbitrary, although most of the examples and motivation in the early part of Chapter 3 use logs base 2 or logs base 10.

*Logarithms play a key role in calculus, but many calculus instructors complain that too many students lack appropriate algebraic manipulation skills with logarithms.*

All precalculus textbooks present radioactive decay as an example of exponential decay. Amazingly, the typical precalculus textbook states that if a radioactive isotope has a half-life of $h$, then the amount left at time $t$ will equal $e^{-kt}$ times the amount at time 0, where $k = \frac{\ln 2}{h}$.

A much clearer formulation would state, as this textbook does, that the amount left at time $t$ will equal $2^{-t/h}$ times the amount at time 0. The unnecessary use of $e$ and $\ln 2$ in this context may suggest to students that $e$ and natural logarithms have only contrived and artificial uses, which is not the message that students should receive from their textbook.

Similarly, many precalculus textbooks consider, for example, a colony of bacteria doubling in size every 3 hours, with the textbook then producing the formula $e^{(t \ln 2)/3}$ for the growth factor after $t$ hours. The simpler and natural formula $2^{t/3}$ seems not to be mentioned in such books. This book presents the more natural approach to such issues of exponential growth and decay.

The crucial concepts of $e$ and natural logarithms are introduced in the second half of Chapter 3. Most precalculus textbooks either present no motivation for $e$ or motivate $e$ via continuously compounding interest or through the limit of an indeterminate expression of the form $1^\infty$; these concepts are difficult for students at this level to understand.

*About half of calculus (namely, integration) deals with area, but most precalculus textbooks barely mention the subject.*

Chapter 3 presents a clean and well-motivated approach to $e$ and the natural logarithm. This approach uses the area (intuitively defined) under the curve $y = \frac{1}{x}$, above the $x$-axis, and between the lines $x = 1$ and $x = c$.

A similar approach to $e$ and the natural logarithm is common in calculus courses. However, this approach is not usually adopted in precalculus textbooks. Using obvious properties of area, the simple presentation given here shows how these ideas can come through clearly without the technicalities of calculus or the messy notation of Riemann sums. Indeed, this precalculus approach to the exponential function and the natural logarithm shows that a good understanding of these subjects need not wait until the calculus course. Students who have seen the approach given here should be well prepared to deal with these concepts in their calculus courses.

The approach taken here also has the advantage that it easily leads, as shown in Chapter 3, to the approximation $\ln(1 + h) \approx h$ for small values of $h$. Furthermore, the same methods show that if $r$ is any number, then $\left(1 + \frac{r}{x}\right)^x \approx e^r$ for large values of $x$. A final bonus of this approach is that the connection between continuously compounding interest and $e$ becomes a nice corollary of natural considerations concerning area.

## Trigonometry

*Trigonometry is the hardest part of precalculus for most students.*

This book gives a gentle introduction to trigonometry, making sure that students are comfortable with the unit circle and with radians before defining the trigonometric functions.

Rather than following the practice of most precalculus books of defining six trigonometric functions all at once, this book has a section on the cosine and sine functions. Then the next section introduces the tangent function and finally the secant, cosecant, and cotangent functions. These latter three functions, which are simply the reciprocals of the three key trigonometric functions, add little content or understanding; thus they do not receive much attention here.

Should the trigonometric functions be introduced via the unit circle or via right triangles? Calculus requires the unit-circle approach because, for example, discussing the Taylor series for $\cos x$ requires us to consider negative values of $x$ and values of $x$ that are more than $\frac{\pi}{2}$ radians. Thus this textbook uses the unit-circle approach, but quickly gives applications to right triangles. The unit-circle approach also allows for a well-motivated introduction to radians.

Most precalculus textbooks define the trigonometric functions using four symbols: $\theta$ or $t$ for the angle and $P(x, y)$ for the endpoint of the radius of the unit circle corresponding to that angle. Why is that endpoint usually called $P(x, y)$ instead of simply $(x, y)$? Even better than just dispensing with $P$, the symbols $x$ and $y$ can also be skipped by denoting the coordinates of the endpoint of the radius as $(\cos\theta, \sin\theta)$, thus defining the cosine and sine. The standard approach of defining $\cos\theta = x$ and $\sin\theta = y$ causes problems when students get to calculus and need to deal with $\cos x$. If students have memorized the notion that cosine is the $x$-coordinate, then they will be thinking that $\cos x$ is the $x$-coordinate of ... oops, this is two different uses of $x$. To avoid the confusion discussed above, this book uses only one symbol to define the trigonometric functions.

## What's New in this Third Edition

- The chapter on systems of linear equations from the previous edition has been eliminated, as has the appendix on parametric curves. Both these items, which deal with topics that are not needed for first-semester calculus, are available as electronic supplements. They are also available in my *Algebra and Trigonometry* book.
- The section on transformations of trigonometric functions has been moved to Chapter 5.
- What are now Chapters 6 and 7 were in the reverse order in the previous edition. Chapter 7 has a new title.
- The main text font has been changed from Lucida to URW Palladio, which is a legal clone of Palatino. The math fonts have been changed from various versions of Lucida to various versions of URW Palladio, Pazo Math, and Computer Modern.

*For more information on the typesetting of this book, see the Colophon at the end of the book.*

- The paper length has been slightly expanded by three-eighths of an inch.
- The new fonts and new page size mean new page breaks and new line breaks. LaTeX handles line breaks well. However, I had to do extensive rewriting to make page breaks come out well. For example, students almost always have an entire Example visible without turning a page.

*The content changes and format changes result in a book that is about one hundred pages shorter than the previous version.*

- Each full page of text now contains at least two marginal notes, as compared to at least one marginal note in the previous edition. A figure or photo counts as a marginal note. When a figure or photo has a caption, the caption does not count as an additional marginal note. The word Example does not count as a marginal note.
- Eighteen new photos relevant to the content have been added.
- A new color scheme has been implemented. Definition boxes are now yellow and result boxes are now blue. Example lines are now orange, and example labels are now white inside orange.
- Definition boxes, result boxes, learning objectives boxes, and example label boxes now have rounded corners for a gentler look.
- Definition boxes and result boxes now have their titles in a darker-shaded sub-box for a catchy appearance.
- Numerous improvements have been made throughout the text based upon suggestions from faculty and students who used the previous edition.
- New exercises have been added in almost all sections. The Appendix now includes worked-out solutions to the Appendix's exercises.

## Comments Welcome

I seek your help in making this a better book. Please send me your comments and your suggestions for improvements. Thanks!

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San Francisco State University

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web site: precalculus.axler.net

# *Acknowledgments*

As usual in a textbook, little attempt has been made to provide proper credit to the original creators of the ideas presented in this book. Where possible, I have tried to improve on standard approaches to this material. However, the absence of a reference does not imply originality on my part. I thank the many mathematicians who have created and refined our beautiful subject.

*Most of the results in this book belong to the common heritage of mathematics, created over thousands of years by clever and curious people.*

Like most mathematicians, I owe huge thanks to Donald Knuth, who invented TeX, and to Leslie Lamport, who invented LaTeX, which I used to typeset this book. I am grateful to the authors of the many open-source LaTeX packages I used to improve the appearance of the book, especially to Hàn Thế Thành for pdfLaTeX, Robert Schlicht for microtype, Frank Mittelbach for multicol, and Till Tantau for Ti*k*Z.

Many thanks also to Wolfram Research for producing *Mathematica*, which is the software I used to create the graphics in this book. I am also grateful to Szabolcs Horvát for the *Mathematica* package MaTeX, which allowed me to place LaTeX-generated labels in the *Mathematica* figures.

The many instructors and students who used the first two editions of this book provided wonderfully useful feedback—thank you!

Several reviewers gave me excellent suggestions as revisions progressed through various stages of development for both the second and third editions of this book. I am grateful to all the reviewers whose names are listed on the following page.

I chose Wiley as the publisher of this book because of the company's commitment to excellence. The people at Wiley have made outstanding contributions to this project, providing wise editorial advice, superb design expertise, high-level production skill, and insightful marketing savvy. I am truly grateful to the following Wiley folks, all of whom helped make this edition a better book than it would have been otherwise: Jennifer Brady, Joanna Dingle, Maureen Eide, John LaVacca III, Giana Milazzo, Ashley Patterson, Mary Ann Price, Laurie Rosatone.

Jen Blue, the accuracy checker, and Katrina Avery, the copy editor, excelled at catching my mathematical and linguistic mistakes.

My awesome partner Carrie Heeter deserves considerable credit for her astute advice and continual encouragement throughout the long book-writing process.

**Many thanks to all of you!**

Sheldon

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# *Preface to the Student*

This book will help prepare you to succeed in calculus. If you master the material in this book, you will have the knowledge, the understanding, and the skills needed to do well in a calculus course.

To learn this material well, you will need to spend serious time reading this book. You cannot expect to absorb mathematics the way you devour a novel. If you read through a section of this book in less than an hour, then you are going too fast. You should pause to ponder and internalize each definition, often by trying to invent some examples in addition to those given in the book. When steps in a calculation are left out in the book, you need to supply the missing pieces, which will require some writing on your part. These activities can be difficult when attempted alone; try to work with a group of a few other students.

Boxes in this book are color-coded to show their function. Specifically, boxes with yellow shading give definitions, and boxes with blue shading give results (which in many books are called theorems, corollaries, etc.).

You will need to spend several hours per section doing the exercises and problems. Make sure that you can do all the exercises and most of the problems, not just the ones assigned for homework. By the way, the difference between an exercise and a problem in this book is that each exercise has a unique correct answer that is a mathematical object such as a number or a function. In contrast, the solutions to problems consist of explanations or examples; thus problems have multiple correct answers.

*Worked-out solutions to the odd-numbered exercises are given at the end of each section.*

*The symbol*  *appears with exercises and problems that require a calculator.*

Have fun, and best wishes in your studies!

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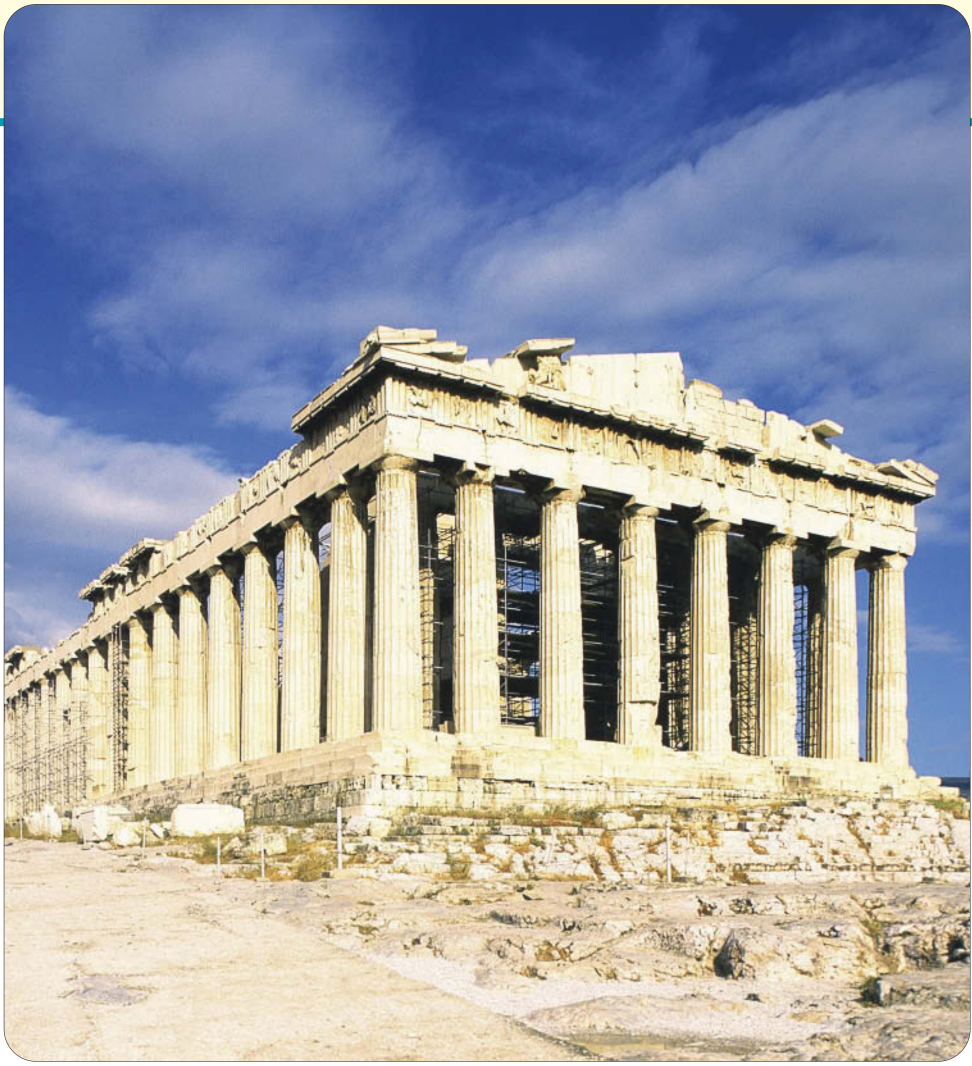
Chapter 0



*The Parthenon, built in Athens over 2400 years ago. The ancient Greeks developed and used remarkably sophisticated mathematics.*

# The Real Numbers

Success in this course will require a good understanding of the basic properties of the real number system. Thus this book begins with a review of the real numbers. This chapter is labeled Chapter 0 to emphasize its review nature.

The first section of this chapter starts with the construction of the real line. This section contains as an optional highlight the ancient Greek proof that no rational number has a square equal to 2. This beautiful result appears here because everyone should see it at least once.

Although this chapter will be mostly review, a thorough grounding in the real number system will serve you well throughout this course. You will need good algebraic manipulation skills. Thus the second section of this chapter reviews fundamental algebra of the real numbers. You will also need to feel comfortable working with inequalities and absolute values, which are reviewed in the last section of this chapter.

Even if your instructor decides to skip this chapter, you may want to read through it. Make sure that you can do the exercises.

## 0.1 The Real Line

**Learning Objectives**

By the end of this section you should be able to

- explain the correspondence between the system of real numbers and the real line;
- show that some real numbers are not rational.

The **integers** are the numbers

$$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots;$$

here the dots indicate that the numbers continue without end in each direction. The sum, difference, and product of any two integers are also integers.

The quotient of two integers is not necessarily an integer. Thus we extend arithmetic to the **rational numbers**, which are numbers of the form

*The use of a horizontal bar to separate the numerator and denominator of a fraction was introduced by Arabian mathematicians about 900 years ago.*

$$\frac{m}{n},$$

where $m$ and $n$ are integers and $n \neq 0$.

Division is the inverse of multiplication, in the sense that we want the equation

$$\frac{m}{n} \cdot n = m$$

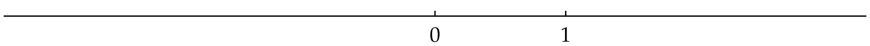
to hold. In the equation above, if we take $n = 0$ and (for example) $m = 1$, we get the nonsensical equation $\frac{1}{0} \cdot 0 = 1$. This equation is nonsensical because multiplying anything by 0 should give 0, not 1. To get around this problem, we leave expressions such as $\frac{1}{0}$ undefined. In other words, division by 0 is prohibited.

The rational numbers form a terrifically useful system. We can add, multiply, subtract, and divide rational numbers (with the exception of division by 0) and stay within the system of rational numbers. Rational numbers suffice for all actual physical measurements, such as length and weight, of any desired accuracy.

However, geometry, algebra, and calculus force us to consider an even richer system of numbers—the real numbers. To see why we need to go beyond the rational numbers, we will investigate the real line.

### Construction of the Real Line

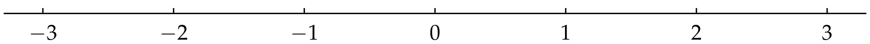
Imagine a horizontal line, extending without end in both directions. Pick a point on this line and label it 0. Pick another point to the right of 0 and label it 1, as in the figure below.




*Two key points on the real line.*

*The symbol for zero was invented in India more than 1100 years ago.*

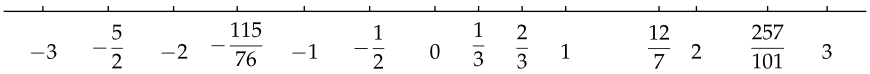
Once the points 0 and 1 have been chosen on the line, everything else is determined by thinking of the distance between 0 and 1 as one unit of length. For example, 2 is one unit to the right of 1. Then 3 is one unit to the right of 2, and so on. The negative integers correspond to moving to the left of 0. Thus $-1$ is one unit to the left of 0. Then $-2$ is one unit to the left of $-1$, and so on.




*Integers on the real line.*

If $n$ is a positive integer, then $\frac{1}{n}$ is to the right of 0 by the length obtained by dividing the segment from 0 to 1 into $n$ segments of equal length. Then $\frac{2}{n}$ is to the right of $\frac{1}{n}$ by the same length, and $\frac{3}{n}$ is to the right of $\frac{2}{n}$ by the same length again, and so on. The negative rational numbers are placed on the line similarly, but to the left of 0.

In this way, every rational number is associated with a point on the line. No figure can show the labels of all the rational numbers, because we can include only finitely many labels in a figure. The figure below shows the line with labels attached to a few of the points corresponding to rational numbers.




*Some rational numbers on the real line.*

We will use the intuitive notion that the line has no gaps and that every conceivable distance can be represented by a point on the line. With these concepts in mind, we call the line shown above the **real line**.

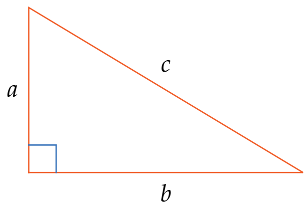
We think of each point on the real line as corresponding to what we call a **real number**. The undefined intuitive notions (such as "no gaps") can be made precise using more advanced mathematics. In this book, we let our intuitive notions of the real line serve to define the system of real numbers.

## Is Every Real Number Rational?

We know that every rational number corresponds to some point on the real line. Does every point on the real line correspond to some rational number? In other words, is every real number rational?

Probably the first people to ponder these issues thought that the rational numbers fill up the entire real line. However, the ancient Greeks discovered that this is not true. To see how they came to this conclusion, we make a brief detour into geometry.

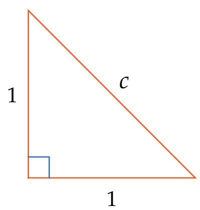
Recall that for each right triangle, the sum of the squares of the lengths of the two sides that form the right angle equals the square of the length of the hypotenuse. The next figure illustrates this result, which is called the Pythagorean Theorem.




*The Pythagorean Theorem for right triangles: $c^2 = a^2 + b^2$.*

*Pythagoras explaining his work (from* THE SCHOOL OF ATHENS, *painted by Raphael around 1510).*

Now consider the special case where both sides that form the right angle have length 1, as in the figure below. In this case, the Pythagorean Theorem states that the length $c$ of the hypotenuse satisfies the equation $c^2 = 2$.




*An isosceles right triangle. The Pythagorean Theorem implies that $c^2 = 2$.*

*The Pythagorean Theorem is named in honor of the Greek mathematician and philosopher Pythagoras, who lived over 2500 years ago. The Babylonians had discovered this result a thousand years before Pythagoras.*

We have just seen that there is a positive real number $c$ such that $c^2 = 2$. This raises the question of whether there exists a rational number $c$ such that $c^2 = 2$.

We could try to find a rational number whose square equals 2 by experimentation. One striking example is

$$\left(\frac{99}{70}\right)^2 = \frac{9801}{4900};$$

here the numerator of the right side misses being twice the denominator by only 1. Although $\left(\frac{99}{70}\right)^2$ is close to 2, it is not exactly equal to 2.

Another example is $\frac{9369319}{6625109}$. The square of this rational number is approximately 1.999999999999977, which is very close to 2 but again is not exactly what we seek.

Because we have found rational numbers whose squares are very close to 2, you might suspect that with further cleverness we could find a rational number whose square equals 2. However, the ancient Greeks proved this is impossible.

This course does not focus much on proofs. However, the Greek proof that there is no rational number whose square equals 2 is one of the great intellectual achievements of humanity. It should be experienced by every educated person. Thus this proof is presented below for your enrichment.

What follows is a proof by contradiction. We will start by assuming that there is a rational number whose square equals 2. Using that assumption, we will arrive at a contradiction. So our assumption must have been incorrect. Thus there is no rational number whose square equals 2.

*Understanding the logical pattern of thinking that goes into this proof can be a valuable asset in dealing with complex issues.*

### *No rational number has a square equal to* 2.

Proof: Suppose there exist integers $m$ and $n$ such that

$$\left(\frac{m}{n}\right)^2 = 2.$$

By canceling any common factors, we can choose $m$ and $n$ to have no factors in common. In other words, $\frac{m}{n}$ is reduced to lowest terms.

The equation above is equivalent to the equation

$$m^2 = 2n^2.$$

This implies that $m^2$ is even; hence $m$ is even (because the square of each odd number is odd). Thus $m = 2k$ for some integer $k$. Substituting $2k$ for $m$ in the equation above gives

$$4k^2 = 2n^2,$$

or equivalently

$$2k^2 = n^2.$$

This implies that $n^2$ is even; hence $n$ is even.

We have now shown that both $m$ and $n$ are even, contradicting our choice of $m$ and $n$ as having no factors in common.

This contradiction means our original assumption that there is a rational number whose square equals 2 must be incorrect. Thus there do not exist integers $m$ and $n$ such that $\left(\frac{m}{n}\right)^2 = 2$.

*"When you have excluded the impossible, whatever remains, however improbable, must be the truth."*
—Sherlock Holmes

The notation $\sqrt{2}$ is used to denote the positive real number $c$ such that $c^2 = 2$. As we saw earlier, the Pythagorean Theorem implies that there exists a real number $\sqrt{2}$ with the property that

$$(\sqrt{2})^2 = 2.$$

The result above implies that $\sqrt{2}$ is not a rational number. Thus not every real number is a rational number. In other words, not every point on the real line corresponds to a rational number.